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Theory of non-Abelian superfluid dynamics

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We write down a theory for non-Abelian superfluids with a partially broken (semisimple) Lie group. We adapt the off-shell formalism of hydrodynamics to superfluids and use it to comment on the superfluid transport compatible with the second law of thermodynamics. We find that the second law can be also used to derive the Josephson equation, which governs dynamics of the Goldstone modes. In the course of our analysis, we derive an alternate and mutually distinct parametrization of the recently proposed classification of hydrodynamic transport and generalize it to superfluids.

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I. INTRODUCTION

Hydrodynamics is the study of universal low energy fluctuations of a quantum system near thermodynamic equilibrium. Any quantum system in this regime, called a *fluid*, can be characterized by a set of transport coefficients such as pressure, viscosity and conductivity. When a part of the global symmetry of the microscopic theory is spontaneously broken in the ground state, low energy fluctuations can also contain massless Goldstone modes [1] corresponding to the broken symmetry. Therefore, the associated fluid, commonly known as a *superfluid* [2–4], contains many new transport coefficients in its spectrum. Superfluidity with a broken U(1) was first observed in liquid ^{4}He [5,6], which since then has been well explored in the literature, at least up to first order in derivatives (see e.g. Refs. [7,8]). In recent years, non-Abelian superfluids have also started to attract some attention (see Ref. [9] and references therein) in relation to p-wave superfluidity observed in liquid ^{3}He [10,11].

The goal of this paper is to set up a theory for superfluids with an arbitrarily broken internal symmetry, which has not been formalized before, and explore the constraints imposed upon it by the second law of thermodynamics. We will also show how a mild modification to the local statement of the second law leads to a derivation of the Josephson equation, which governs dynamics of the Goldstone modes. This equation has been imposed by hand in most previous treatments of superfluids. While addressing these questions, we will propose a natural and mutually distinct classification of the entire (super)fluid transport, which in the ordinary fluid limit gives a refined parametrization of the classification proposed by Refs. [12,13].

II. SPONTANEOUS SYMMETRY BREAKING

Let us start with a quick recap of spontaneous symmetry breaking; details can be found in Sec. 19 of Ref. [14]. Consider a microscopic theory invariant under spacetime translations and action of a spacetime invariant semisimple Lie group G (with Lie algebra $i\mathfrak{g}$). Let ψ be a field in the theory transforming under some unitary representation $\mathcal{D}(G)$ of G, i.e. under a $g \in G$ transformation $\psi \to \mathcal{D}(g)\psi$. ψ is said to spontaneously break the symmetry from G to its Lie subgroup $H \subset G$ (with Lie subalgebra $i\mathfrak{h} \subset i\mathfrak{g}$), if its ground state expectation value $\langle\psi\rangle$ is only invariant under H; i.e. $\mathcal{D}(h)\langle\psi\rangle = \langle\psi\rangle$ if and only if $h \in H$. $\mathcal{D}(g)\langle\psi\rangle$ with $g \notin H$ are "other" ground states the system could have spontaneously chosen from. Around $\langle\psi\rangle$, the field ψ can be expressed as group transformation of a reference field $\tilde{\psi}$, i.e. $\psi = \mathcal{D}(\gamma)\tilde{\psi}$, defined by

$$\tilde{\psi}^\dagger \mathcal{D}(X)\langle\psi\rangle = \tilde{\psi}^\dagger\langle\psi\rangle, \quad \forall\ X \in \mathfrak{g}. \tag{1}$$

Roughly speaking, γ corresponds to fluctuations of ψ which takes us to the nearby ground states with no energy cost, while $\tilde{\psi}$ contains genuine excitations of ψ. Note that Eq. (1) is invariant under $\tilde{\psi} \to \mathcal{D}(h)\tilde{\psi}$ with $h \in H$ and hence determines γ only up to a coset equivalence $\gamma \sim \gamma h$. Let us pick a representative from each coset $\gamma = \gamma(\varphi)$ parametrized by a field φ living in the Lie algebra quotient $\mathfrak{g}/\mathfrak{h}$, which can be identified as the *Goldstone modes* of the broken symmetry. Under a $g \in G$ transformation,

$$\gamma(\varphi) \to g\gamma(\varphi)h(\varphi, g)^{-1}, \qquad \tilde{\psi} \to \mathcal{D}(h(\varphi, g))\tilde{\psi}, \tag{2}$$

for some $h(\varphi, g) \in H$, such that $\psi \to \mathcal{D}(g)\psi$ and Eq. (1) remains invariant. From these transformation properties, it is clear that the theory cannot contain a mass term for φ, rendering it massless. It follows that φ substantially affects the low energy fluctuations of the theory and must be taken into account in the superfluid description. A quick comparison can be made with the Abelian case, where $G = \mathrm{U}(1)$ is broken down to $H = \{1\}$, with $\gamma(\varphi) = \mathrm{e}^{-i\varphi}$. Under a $\mathrm{e}^{i\Lambda} \in \mathrm{U}(1)$ transformation, $\varphi \to \varphi - \Lambda$, which is well known for Abelian symmetry breaking.

For further analysis, it is helpful to introduce a set of generators $\{\mathrm{t}_\alpha\} = \{\mathrm{t}_i, \mathrm{t}_a\}$ of G such that the subset $\{\mathrm{t}_i\}$

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generates H. We normalize these generators by choosing $\mathrm{t}_\alpha \cdot \mathrm{t}_\beta = 2\mathrm{Tr}[\mathrm{t}_\alpha \mathrm{t}_\beta] = \eta_{\alpha\beta}$, where $\eta_{\alpha\beta}$ is a diagonal matrix with entries ± 1. We define the action of a $g \in G$ on these generators as $\mathrm{t}_\alpha \to \mathrm{Ad}_g(\mathrm{t}_\alpha) = (\mathrm{Ad}_g)^\beta{}_\alpha \mathrm{t}_\beta = g \mathrm{t}_\alpha g^{-1}$.

While dealing with partially broken symmetries, we are confronted with an obstacle: the quotient $\mathfrak{g}/\mathfrak{h}$ is not a Lie Algebra and hence φ does not transform "nicely" under the action of G, which poses a difficulty while formulating superfluids. We find that this problem can be circumvented by introducing a pair of projection operators $\mathrm{P}, \bar{\mathrm{P}}\colon\ \mathfrak{g} \to \mathfrak{g}$ defined as

$$\begin{aligned} \mathrm{P}(\mathrm{t}_\alpha) &= \mathrm{P}^\beta{}_\alpha \mathrm{t}_\beta = ((\mathrm{Ad}_\gamma)^\beta{}_i (\mathrm{Ad}_{\gamma^{-1}})^i{}_\alpha) \mathrm{t}_\beta, \\ \bar{\mathrm{P}}(\mathrm{t}_\alpha) &= \bar{\mathrm{P}}^\beta{}_\alpha \mathrm{t}_\beta = (\delta^\beta{}_\alpha - \mathrm{P}^\beta{}_\alpha) \mathrm{t}_\beta. \end{aligned} \tag{3}$$

They covariantly project out components of $X \in \mathfrak{g}$ along or against the residual symmetry respectively. Using $\bar{\mathrm{P}}$, we can rebundle the information in φ into a "covariant derivative" $\tilde{\partial}_\mu \varphi = \bar{\mathrm{P}}(i\partial_\mu \gamma(\varphi)\gamma(\varphi)^{-1}) \in \mathfrak{g}$. Alternatively, we can use the Maurer-Cartan form on G/H to define this covariant derivative. Introducing the operators P, $\bar{\mathrm{P}}$, however, will also simplify the notation in the following non-Abelian superfluid analysis, resulting in a pleasant resemblance with the better known Abelian results. Additionally, we can revert back to the ordinary fluids at any point by setting $\bar{\mathrm{P}} = 0$, $\mathrm{P} = \mathrm{id}_\mathfrak{g}$ (identity in $\mathfrak{g}$).

A. Example

Consider the group $G = SU(2)$ generated by $\{\mathrm{t}_\alpha = \frac{1}{2}\sigma_\alpha\}$, where σ_α are the Pauli matrices, which is broken down to $H = \mathrm{U}(1)$ spanned by $\{\mathrm{t}_1\}$. A convenient choice of Goldstone modes is $\gamma(\varphi) = \mathrm{e}^{-i\varphi^2 \mathrm{t}_2} \mathrm{e}^{-i\varphi^3 \mathrm{t}_3}$. Under an infinitesimal transformation $g = \mathrm{e}^{i\Lambda^a \mathrm{t}_a} \in SU(2)$, the Goldstone modes transform noncovariantly,

$$\begin{aligned} \varphi^2 &\to \varphi^2 - \Lambda^2 + \tan\varphi^3 (\Lambda^1 \cos\varphi^2 - \Lambda^3 \sin\varphi^2), \\ \varphi^3 &\to \varphi^3 - \Lambda^3 \cos\varphi^2 - \Lambda^1 \sin\varphi^2, \end{aligned} \tag{4}$$

with $h(\varphi, g) = \mathrm{e}^{i \sec\varphi^3 (\Lambda^1 \cos\varphi^2 - \Lambda^3 \sin\varphi^2) \mathrm{t}_1}$. On the other hand, we can check that the covariant derivative

$$\tilde{\partial}_\mu \varphi = \begin{pmatrix} -\frac{1}{2}\sin(2\varphi^3)\cos\varphi^2 \partial_\mu \varphi^2 + \sin\varphi_2 \partial_\mu \varphi^3 \\ \cos^2\varphi^3 \partial_\mu \varphi^2 \\ \frac{1}{2}\sin(2\varphi^3)\sin\varphi^2 \partial_\mu \varphi^2 + \cos\varphi_2 \partial_\mu \varphi^3 \end{pmatrix} \tag{5}$$

transforms covariantly, i.e. $\tilde{\partial}_\mu \varphi \to \mathrm{Ad}_g(\tilde{\partial}_\mu \varphi)$.

III. SUPERFLUID DYNAMICS

We are interested in studying low energy fluctuations of a theory with a spontaneously broken internal symmetry. As eluded before, any such description must contain the Goldstone modes φ as a dynamical field, with dynamics provided by some $\dim(\mathfrak{g}/\mathfrak{h})$-component equation,

$$K = 0 \in \bar{\mathrm{P}}(\mathfrak{g}). \tag{6}$$

Allowing for an arbitrary dynamical equation for φ is a novel feature of our formalism, which in the conventional treatment of superfluids is taken to be the "Josephson equation" by hand (see e.g. Ref. [7]). An exception to this is Ref. [15], where authors derive the equilibrium version of the Josephson equation using an effective action. Here, however, we will show that it follows naturally by imposing the second law of thermodynamics.

A theory invariant under spacetime translations and G transformations must also contain an associated conserved *energy-momentum tensor* $T^{\mu\nu}$ and a $\mathfrak{g}$-valued *charge current* J^μ in its spectrum. To probe these observables, we couple the theory to a slowly varying metric $g_{\mu\nu}$ and a gauge field A_μ. The covariant derivative associated with the Levi-Cività connection $\Gamma^\lambda{}_{\mu\nu}$ is denoted by ∇_μ, while the one associated with A_μ and $\Gamma^\lambda{}_{\mu\nu}$ is denoted by D_μ. In the presence of these external sources, respective conservation laws are

$$\nabla_\nu T^{\nu\mu} = F^{\mu\nu} \cdot J_\nu + \xi^\mu \cdot K + \mathrm{T}^{\mu\perp}_{\mathrm{H}}, \qquad \mathrm{D}_\mu J^\mu = \mathrm{J}^\perp_{\mathrm{H}} - K, \tag{7}$$

where we have allowed for φ to go off shell ($K \neq 0$). $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]} - i[A_\mu, A_\nu] \in \mathfrak{g}$ is the gauge field strength, and $\xi_\mu = \bar{\mathrm{P}}(A_\mu) + \tilde{\partial}_\mu \varphi \in \bar{\mathrm{P}}(\mathfrak{g})$ is called the *superfluid velocity*. The Hall currents $\mathrm{T}^{\mu\perp}_{\mathrm{H}}$ and $\mathrm{J}^\perp_{\mathrm{H}}$ represent the contribution from possible gravitational and flavor anomalies in the microscopic theory respectively. One way to derive the conservation laws (7) is to consider a field theory effective action $S[g_{\mu\nu}, A_\mu, \varphi]$ and parametrize its infinitesimal variation as

$$\delta S = \int \{\mathrm{d}x^\mu\} \sqrt{-g} \left[\frac{1}{2} T^{\mu\nu} \delta g_{\mu\nu} + J^\mu \cdot \delta A_\mu + K \cdot \tilde{\delta}\varphi\right], \tag{8}$$

where $g = \det g_{\mu\nu}$ and $\tilde{\delta}\varphi = \bar{\mathrm{P}}(i\delta\gamma(\varphi)\gamma(\varphi)^{-1})$. Given this setup, one can check that the conservation laws (7) are merely the Ward identities corresponding to infinitesimal diffeomorphisms and G gauge transformations.

The conservation laws (7) can provide dynamics for a theory formulated in terms of the *hydrodynamic fields*: normalized 4-velocity u^μ (with $u^\mu u_\mu = -1$), temperature T, and chemical potential $\mu \in \mathfrak{g}$, in addition to the Goldstone modes φ. It should be noted, however, that these are merely some fields chosen to describe the system and, like in any field theory, can admit an arbitrary redefinition; we will return to this issue later.

In general, the observables $T^{\mu\nu}$, J^μ, and K appearing in Eqs. (6) and (7) can have an arbitrary dependence on the fields $\Psi = \{u^\mu, T, \mu, g_{\mu\nu}, A_\mu, \xi_\mu\}$. In hydrodynamics, however, we are only interested in the low energy fluctuations of the constituent fields Ψ, which can be translated as the configurations of Ψ that admit a perturbative expansion in derivatives. This allows us to write down the most generic allowed expressions for $T^{\mu\nu}$, J^μ, and K in terms of Ψ truncated up to a finite order in derivatives, called the *superfluid constitutive relations*. At a given order, constitutive relations will contain all the possible tensor structures allowed by symmetry (modulo field redefinitions) called *data*, multiplied with arbitrary scalars called *transport coefficients*. The explicit functional form of these transport coefficients depends on the underlying microscopic theory, but we can put some stringent constraints on them by imposing some physical requirements such as the local second law of thermodynamics:

Given a set of constitutive relations $T^{\mu\nu}$, J^μ, and K, there must exist an entropy current J^μ_S of which the divergence is non-negative, i.e. $\nabla_\mu J^\mu_S \geq 0$, for all the superfluid configurations satisfying the conservation laws (7).

It is worth pointing out that this statement is slightly stronger than the one used previously in the superfluid literature (e.g. Ref. [7]), as it is imposed even when φ is off shell. This extra information fixes Eq. (6) to be the Josephson equation, as we will now illustrate.

A. Ideal superfluids

Consider the most generic constitutive relations and entropy current of a superfluid at zero derivative order,

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu} + \xi^\mu \cdot \rho_s \cdot \xi^\nu,$$
$$J^\mu = q u^\mu + q_s \cdot \xi^\mu, \qquad J^\mu_S = s u^\mu + s_s \cdot \xi^\mu, \tag{9}$$

along with a scalar K. We have fixed the ideal order definition of u^μ by eliminating a term like $\epsilon_s \cdot \xi^{(\mu} u^{\nu)}$ from $T^{\mu\nu}$. On the other hand, ideal order definitions of T, μ are fixed via the *first law of thermodynamics*,

$$\mathrm{d}\epsilon = T\mathrm{d}s + \mu^\alpha \mathrm{D} q_\alpha + \frac{1}{2} f_{\alpha\beta} \mathrm{D}(\xi^{\mu,\alpha}\xi^\beta_\mu), \tag{10}$$

where we have defined $f \in \bar{\mathrm{P}}(\mathfrak{g}) \times_{\mathrm{sym}} \bar{\mathrm{P}}(\mathfrak{g})$ (i.e. $f^{\alpha\beta} = f^{\beta\alpha}$ and $\mathrm{P}^\gamma{}_\alpha f^{\alpha\beta} = 0$). Using the conservation laws (7) and imposing $\nabla_\mu J^\mu_S \geq 0$, we can find the following constraints,

$$\epsilon = sT + q\cdot\mu - P \qquad (\text{Euler relation}),$$
$$K = -\frac{1}{T}\alpha \cdot (u^\mu \xi_\mu - \bar{\mathrm{P}}(\mu)) + \mathrm{D}_\mu(f\cdot\xi^\mu) + i[\xi_\mu, f\cdot\xi^\mu],$$
$$s_s = 0, \qquad \rho_s = -q_s = f, \qquad \mu \cdot i[\xi_\mu, f\cdot\xi^\mu] = 0, \tag{11}$$

for some $\alpha \in \bar{\mathrm{P}}(\mathfrak{g}) \times \bar{\mathrm{P}}(\mathfrak{g})$ with positive eigenvalues. Plugging these back in, we get the constitutive relations of an ideal non-Abelian superfluid. The surviving coefficients can be interpreted as *pressure* P, *energy density* ϵ, *charge density* q, *entropy density* s, and *superfluid density* f. Setting $K = 0$, we get the Josephson equation extended to ideal non-Abelian superfluids,

$$u^\mu \xi_\mu = \bar{\mathrm{P}}(\mu) + T\alpha' \cdot (\mathrm{D}_\mu(f\cdot\xi^\mu) + i[\xi_\mu, f\cdot\xi^\mu]), \tag{12}$$

where $\alpha' \in \bar{\mathrm{P}}(\mathfrak{g})$ is defined via $\alpha'\cdot\alpha = \bar{\mathrm{P}}$. It says that a change in φ in the direction of the flow is given by the chemical potential μ, supplemented with some derivative corrections. In equilibrium, Eq. (12) reduces to $\mathrm{D}_\mu(f\cdot\xi^\mu) + i[\xi_\mu, f\cdot\xi^\mu] = 0$, which was obtained by Ref. [15] in the Abelian case using an equilibrium effective action.

Finally, α can be interpreted as a first order transport coefficient matrix, which contributes toward the production of entropy through α',

$$\nabla_\mu J^\mu_S = (\mathrm{D}_\mu(f\cdot\xi^\mu) + \cdots)\cdot\alpha'\cdot(\mathrm{D}_\mu(f\cdot\xi^\mu) + \cdots). \tag{13}$$

Interestingly, we see that ideal superfluids, unlike ordinary fluids, can cause dissipation but with the effect being a higher derivative can be ignored at ideal order.

IV. OFF-SHELL FORMALISM FOR SUPERFLUIDS

Having worked out ideal superfluids, we can in principle extend this procedure to constitutive relations with an arbitrarily high number of derivatives. However, implementing the second law becomes messier as we go higher in the derivative expansion, because we are required to recursively implement the lower order conservation laws (see e.g. Ref. [16]). Fortunately, as realized by Ref. [17] for ordinary fluids, it is possible to extend the second law to cases where the conservation laws are not satisfied (i.e. superfluid is kept in contact with an external bath), by adding an arbitrary combination of the conservation laws (7) to $\nabla_\mu J^\mu_S$, giving

$$\nabla_\mu J^\mu_S + \beta_\mu(\nabla_\nu T^{\nu\mu} - F^{\mu\nu}\cdot J_\nu - \xi^\mu\cdot K - \mathrm{T}^{\mu\perp}_{\mathrm{H}})$$
$$+ \nu\cdot(\mathrm{D}_\mu J^\mu + K - \mathrm{J}^\perp_{\mathrm{H}}) \geq 0. \tag{14}$$

Here, β^μ and ν are some arbitrary fields. Let us define $N^\mu = J^\mu_S + \beta_\nu T^{\nu\mu} + \nu\cdot J^\mu$ and $\mathrm{N}^\perp_{\mathrm{H}} = \beta_\mu \mathrm{T}^{\mu\perp}_{\mathrm{H}} + \nu\cdot\mathrm{J}^\perp_{\mathrm{H}}$. In terms of these, Eq. (14) can be recast in a more useful form,

$$\nabla_\mu N^\mu - \mathrm{N}^\perp_{\mathrm{H}} - \Delta = \Phi\cdot\mathcal{C}, \tag{15}$$

where Δ is a positive definite quadratic form. To make the notation compact, we have introduced

$$\mathcal{C} = (T^{\mu\nu} \quad J^\rho \quad K), \qquad \Phi = (\tfrac{1}{2}\delta_{\mathcal{B}} g_{\mu\nu} \quad \delta_{\mathcal{B}} A_\rho \quad \tilde{\delta}_{\mathcal{B}}\varphi), \tag{16}$$

which are vectors in the composite space $\mathfrak{V} = (\text{sym tensor}) \oplus (\mathfrak{g}\times\text{vector}) \oplus \bar{\mathrm{P}}(\mathfrak{g})$. "$\delta_{\mathcal{B}}$" denotes an infinitesimal diffeomorphism and gauge transformation with parameters $\mathcal{B} = \{\beta^\mu, \Lambda_\beta = \nu - A_\mu\beta^\mu\}$ respectively.

$$\begin{aligned}\delta_\mathcal{B} g_{\mu\nu} &= \pounds_\beta g_{\mu\nu} = 2\nabla_{(\mu}\beta_{\nu)},\\ \delta_\mathcal{B} A_\mu &= \pounds_\beta A_\mu + \partial_\mu\Lambda_\beta - i[A_\mu, \Lambda_\beta] = \mathrm{D}_\mu\nu + \beta^\nu F_{\mu\nu},\\ \tilde{\delta}_\mathcal{B}\varphi &= \bar{\mathrm{P}}(i\delta_\mathcal{B}\gamma(\varphi)\gamma(\varphi)^{-1})\\ &= \bar{\mathrm{P}}(i\pounds_\beta\gamma(\varphi)\gamma(\varphi)^{-1} + \Lambda_\beta) = \beta^\mu\xi_\mu - \bar{\mathrm{P}}(\nu).\end{aligned}$$

One can check that the ideal order definitions of u^μ, T, and μ [given around Eq. (10)] imply the relations $\beta^\mu = u^\mu/T$, $\nu = \mu/T$ at ideal order. We fix the remaining ambiguity in the fluid fields by assuming these relations to hold at all orders in the derivative expansion. Having done that, the allowed superfluid constitutive relations are the most generic expressions for $T^{\mu\nu}$, J^μ, and K in terms of Ψ which satisfy Eq. (15) for some N^μ and $\Delta \geq 0$.

Note that it is always possible to write down terms $N^\mu_\mathrm{S} \in N^\mu$ of which the divergence is either zero or is balanced by some counterterms $\Delta_\mathrm{S} \in \Delta$, i.e $\nabla_\mu N^\mu_\mathrm{S} = \Delta_\mathrm{S}$. We refer to these terms as class S. They are not genuine (super)fluid transport; instead, they parametrize the multitude of entropy currents which satisfy the second law for the same set of constitutive relations.

We split the tensor structures that can appear in the constitutive relations into two sectors: "nonhydrostatic data" (independent data that contain at least one instance of "$\delta_\mathcal{B}$") and "hydrostatic data" (the largest collection of independent data with no nonhydrostatic linear combination). The second law, similar to the known results in ordinary fluids [18,19], imposes strict equality constraints in the hydrostatic sector, while in the nonhydrostatic sector, it only gives a few inequalities at the first order in derivatives and none thereafter. We will present a quick proof of this statement; in the hydrostatic sector, we will closely follow Ref. [13] with appropriate modifications for superfluids, while in the nonhydrostatic sector, our presentation will be independent and simpler.

A. Hydrostatic sector

Consider the most generic constitutive relations $\mathcal{C} = \mathcal{C}_\mathrm{hydrostatic}$ which are solely made up of the hydrostatic data. For these, every independent term in the rhs of Eq. (15) will contain exactly one bare (which is not acted upon by a derivative) $\delta_\mathcal{B}$. Hence, the associated N^μ also must contain the hydrostatic data only; otherwise, $\nabla_\mu N^\mu$ will either be void of a bare $\delta_\mathcal{B}$ or will contain multiple $\delta_\mathcal{B}$'s. The most generic N^μ in the hydrostatic sector can therefore be written as

$$N^\mu_\mathrm{hydrostatic} = (\mathcal{N}\beta^\mu + \Theta^\mu_\mathcal{N}) + \mathbb{N}^\mu, \tag{17}$$

where $\mathbb{N}^\mu u_\mu = 0$. $\mathcal{N}$ is the most generic scalar made out of the independent hydrostatic data, modulo the total derivative terms. $\Theta^\mu_\mathcal{N}$ is an $\mathcal{N}$ dependent nonhydrostatic vector defined via

$$\nabla_\mu(\mathcal{N}\beta^\mu) = \frac{1}{\sqrt{-g}}\delta_\mathcal{B}(\sqrt{-g}\mathcal{N}) = \Phi\cdot\mathcal{C}_{\mathrm{H}_S} - \nabla_\mu\Theta^\mu_\mathcal{N}, \tag{18}$$

which ensures that $\nabla_\mu(\mathcal{N}\beta^\mu + \Theta^\mu_\mathcal{N})$ has a bare $\delta_\mathcal{B}$. Equation (18) also defines the constitutive relations $\mathcal{C}_{\mathrm{H}_S}$ associated with $\mathcal{N}$, called class H_S. $\mathbb{N}^\mu$ on the other hand is the most generic hydrostatic vector transverse to u^μ, such that $\nabla_\mu\mathbb{N}^\mu - \mathrm{N}^\perp_\mathrm{H}$ has exactly one bare $\delta_\mathcal{B}$. This requirement happens to completely determine $\mathbb{N}^\mu$ up to some constants [20,21], which turns out to be independent of φ and includes the terms responsible for anomalies. Therefore, we can directly import $\mathbb{N}^\mu$ and the respective class $\mathrm{H}_V\cup\mathrm{A}$ constitutive relations $\mathcal{C}_{\mathrm{H}_V} + \mathcal{C}_\mathrm{A}$ from the ordinary fluid literature [13], where class A is the contribution from anomalies. $\mathcal{C}_\mathrm{hydrostatic} = \mathcal{C}_{\mathrm{H}_S} + \mathcal{C}_{\mathrm{H}_V} + \mathcal{C}_\mathrm{A}$ are therefore the most generic hydrostatic constitutive relations compatible with the second law. Comparing these to the most generic expressions allowed by symmetry, we can read out the equality constraints.

It is worth pointing out here that if we focus on *equilibrium* superfluid configurations, where $\mathcal{B} = \{\beta^\mu, \Lambda_\beta\}$ generates an isometry, then the hydrostatic constraints can also be obtained using an effective action [15]:

$$S_\mathrm{eqb} = \int\{\mathrm{d}x^i\}(\sqrt{-g}N^t_\mathrm{hydrostatic})_\mathrm{eqb}. \tag{19}$$

Here, we have chosen coordinates $\{x^\mu\} = \{t, x^i\}$ to set $\beta^\mu = \partial_t$. In this picture, S_eqb can be seen as the most generic scalar on a constant time slice involving $g_{\mu\nu}$, A_μ, and φ. Quite naturally, the equilibrium Josephson equation follows from here by extremizing $\delta S_\mathrm{eqb}/\delta\varphi = 0$.

B. Nonhydrostatic sector

This sector of hydrodynamics contains constitutive relations $\mathcal{C} = \mathcal{C}_\text{non-hydrostatic}$ which are purely made of the nonhydrostatic data. Since all nonhydrostatic data have at least one $\delta_\mathcal{B}$, it can be written as a differential operator acting on Φ defined in Eq. (16). Introducing a symmetric covariant derivative operator $\mathrm{D}^n = \mathrm{D}_{(\mu_1}\ldots\mathrm{D}_{\mu_n)}$ (antisymmetric derivatives can be represented by curvature and field strength), the most generic nonhydrostatic constitutive relations can therefore be written in a compact form,

$$\mathcal{C}_\mathrm{nonhydrostatic} = -\sum_{n=0}^{\infty}\frac{1}{2}[\mathfrak{C}_n\cdot(\mathrm{D}^n\Phi) + \mathrm{D}^n(\mathfrak{C}_n\cdot\Phi)]. \tag{20}$$

$\mathfrak{C}_n \in \mathfrak{B}\times\mathfrak{B}$ are matrices with additional n symmetric indices to be contracted with D^n. The last term in Eq. (20) is taken purely for convenience and can be absorbed into the first via differentiation by parts. Let us factor $\mathcal{C}_\mathrm{nonhydrostatic}$

into a dissipative (class D) and a nondissipative (class $\bar{\mathrm{D}}$) part parametrized by

$$\mathfrak{D}_n = \frac{1}{2}(\mathfrak{C}_n + (-)^n \mathfrak{C}_n^{\mathrm{T}}), \qquad \bar{\mathfrak{D}}_n = \frac{1}{2}(\mathfrak{C}_n - (-)^n \mathfrak{C}_n^{\mathrm{T}}) \tag{21}$$

respectively. The nomenclature can be justified by multiplying Eq. (20) with Φ giving us (see also Refs. [18,19])

$$\Phi \cdot \mathcal{C}_{\mathrm{D}} = -\Delta_{\mathrm{D}} + \nabla_\mu N^\mu_{\mathrm{D}}, \qquad \Phi \cdot \mathcal{C}_{\bar{\mathrm{D}}} = \nabla_\mu N^\mu_{\bar{\mathrm{D}}}, \tag{22}$$

where N^μ_{D}, $N^\mu_{\bar{\mathrm{D}}}$ are some vectors gained via successive differentiation by parts. Δ_{D}, however, is given as

$$\Delta_{\mathrm{D}} = (\Upsilon\Phi) \cdot \mathfrak{D}_0^{(0)} \cdot (\Upsilon\Phi), \tag{23}$$

where $\Upsilon = \sum_{d=0}^{\infty} \Upsilon_d : \mathfrak{B} \to \mathfrak{B}$ is a differential operator defined by [$\mathfrak{D}_0^{(n)}$ is the part of $\mathfrak{D}_0$ with n number of derivatives, and "$\dagger$" denotes the conjugate of a differential operator: $\Phi_1 \cdot (\mathcal{O}\Phi_2) = (\mathcal{O}^\dagger \Phi_1) \cdot \Phi_2 + \nabla_\mu (\cdots)^\mu$]

$$\Upsilon_{d+1}\Big|_{d=1}^{\infty} = -(\mathfrak{D}_0^{(0)})^{-1} \cdot \left[\sum_{k=1}^{d-1} \Upsilon_k^\dagger + \frac{1}{2}\Upsilon_d^\dagger\right](\mathfrak{D}_0^{(0)} \cdot \Upsilon_d),$$
$$\Upsilon_0 = 1, \qquad \Upsilon_1 = \frac{1}{2}(\mathfrak{D}_0^{(0)})^{-1} \cdot \sum_{n=1}^{\infty} (\mathfrak{D}_0^{(n)} + \mathfrak{D}_n \mathrm{D}^n). \tag{24}$$

Comparing Eqs. (22) and (15), we can see that class $\bar{\mathrm{D}}$ constitutive relations satisfy the second law with $N^\mu = N^\mu_{\bar{\mathrm{D}}}$ and $\Delta = 0$, hence the name nondissipative. On the other hand, dissipative class D constitutive relations satisfy the second law with $N^\mu = N^\mu_{\mathrm{D}}$ and $\Delta = \Delta_{\mathrm{D}}$. The condition $\Delta \geq 0$ implies that all the eigenvalues of the zero derivative matrix $\mathfrak{D}_0^{(0)} \in \mathfrak{B} \times \mathfrak{B}$ are non-negative. It follows that the only constraints imposed by the second law in the nonhydrostatic sector are some inequalities in class D at the first order in derivatives.

At the end of the day, we are only interested in describing the superfluid and not its surroundings; hence, the constitutive relations only differing by combinations of the conservation laws must be identified. It can be verified that for the constitutive relations satisfying Eq. (15) the conservation laws (7) are purely nonhydrostatic. Hence, without loss of generality, we can use them to eliminate a vector $u^\mu \delta_{\mathcal{B}} g_{\mu\nu}$ and a $\mathfrak{g}$-valued scalar $u^\mu \delta_{\mathcal{B}} A_\mu$ from the non hydrostatic data. The upshot of this is that we can drop the respective terms from $\mathcal{C}_{\mathrm{D}}$ and $\mathcal{C}_{\bar{\mathrm{D}}}$. Had we eliminated any other data using the conservation laws, the respective constitutive relations would be related to the current ones, at most, by a field redefinition.

V. EXAMPLE: FIRST ORDER SUPERFLUIDS

For an illustration of the off-shell formalism developed above, we briefly outline the (non-Abelian) parity-even superfluids up to first order in derivatives. Let us start with the hydrostatic sector. For class H_S, the scalar $\mathcal{N}$ is made of Lorentz scalars

$$\mathcal{N} = P + \frac{1}{T} f_1^\alpha \xi^\mu_\alpha \partial_\mu T + T f^\alpha_{2\beta} \xi^\mu_\alpha \mathrm{D}_\mu \nu^\beta + f_{3\gamma} \xi^\mu_\alpha \xi^{\nu,\alpha} \mathrm{D}_\mu \xi^\gamma_\nu$$
$$+ (f_5^{[\alpha\beta]} \mathrm{D}_\mu u_\nu + f_{6\gamma}^{[\alpha\beta]} F^\gamma_{\mu\nu} + f_{7\gamma}^{\langle\alpha\beta\rangle} \mathrm{D}_\mu \xi^\gamma_\nu) \xi^\mu_\alpha \xi^\nu_\beta. \tag{25}$$

Here, we have avoided a total derivative term $\mathrm{D}_\mu \xi^\mu_\alpha$, which is the sole member of class S with $N^\mu_{\mathrm{S}} = \nabla_\nu (f_4^\alpha u^{[\mu} \xi^{\nu]}_\alpha)$ and does not contribute to the constitutive relations. The term coupling to $f_{3\gamma}$ can be removed by redefining φ. The remaining coefficients characterize the hydrostatic superfluid transport. The results for classes H_V and A can be taken directly from Ref. [13], which, being parity odd, can be ignored for this example.

Finally, the nonhydrostatic classes D and $\bar{\mathrm{D}}$ at first order are characterized by an ideal order matrix $\mathfrak{C}_0^{(0)}$,

$$\mathfrak{C}_0^{(0)} = \begin{pmatrix} \eta^{\mu\nu\rho\sigma} & \chi^{\mu\nu\rho}_\alpha & x^{\mu\nu}_\alpha \\ \chi'^{\mu\rho\sigma}_\beta & \sigma^{\mu\rho}_{\alpha\beta} & y^\mu_{\alpha\beta} \\ x'^{\rho\sigma}_\beta & y'^\rho_{\alpha\beta} & \alpha_{\alpha\beta} \end{pmatrix}, \tag{26}$$

where the components are the most generic u^μ-transverse tensors written in terms of the fluid and background fields. The antisymmetric part $\bar{\mathfrak{D}}_0^{(0)} = \mathfrak{C}_0^{(0)}|_{\mathrm{assym}}$ characterizes class $\bar{\mathrm{D}}$, while the symmetric part $\mathfrak{D}_0^{(0)} = \mathfrak{C}_0^{(0)}|_{\mathrm{sym}}$ with positive eigenvalues belongs to class D. The respective constitutive relations can be obtained trivially from here using Eqs. (18) and (20).

VI. OUTLOOK

This completes our analysis of the (non-Abelian) superfluid constitutive relations compatible with the second law of thermodynamics. Similar to an ordinary fluid, we find that the second law gives no constraints in the nondissipative nonhydrostatic sector, while it only gives inequalities at the first derivative order in the dissipative sector. In the hydrostatic sector, however, we get equality-type constraints at every derivative order, which can be worked out using an equilibrium effective action. In addition, the second law also gives us the Josephson equation which governs motion of the Goldstone modes corresponding to the broken symmetry.

In the quest of finding the constraints using offshell formalism, we have classified the entire (super)fluid transport into five mutually distinct classes: A, H_S, H_V, D and $\bar{\mathrm{D}}$ along with a class S worth of arbitrariness in the entropy current. This is in contrast with the recent, heavily redundant, classification of ordinary hydrodynamics in Ref. [12]. An added benefit of working in the offshell formalism is that it provides a natural setting to write down an effective action describing (super)fluids. As a prototype, constitutive relations in Class H_S and their dynamical equations can be obtained from an effective action $S_{\mathrm{H}_S} = \int\{\mathrm{d}x^\mu\}\sqrt{-g}\mathcal{N}$ (see Ref. [13] for related details). For the remaining classes, writing down an effective action needs working in the Schwinger-Keldysh formalism [13,22–27], which we leave for future explorations.

In this paper, we concentrate on fluids with broken internal symmetries. The procedure can also be extended to the breaking of spacetime symmetries, interpreted as introducing spacetime boundaries/surfaces in the (super) fluid [28,29]. It will be interesting to see how the second law constrains the surface transport coefficients in (super) fluids and if there is a natural extension of the presented classification to surface transport.

Finally, all of the results presented here can easily be extended to nonrelativistic superfluids using the null fluid formalism of Refs. [30–32]. In a companion paper [33], we use "null superfluids" to work out the constraints on Abelian nonrelativistic superfluid transport up to first order in the derivative expansion.

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